Finding and using exact solutions of the Einstein equations

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Abstract. The evolution of the methods used to find solutions of Einstein's field equations during the last 100 years is described. Early papers used assumptions on the coordinate forms of the metrics. Since the 1950s more invariant methods have been deployed in most new papers. The uses to which the solutions found have been put are discussed, and it is shown that they have played an important role in the development of many aspects, both mathematical and physical, of general relativity.

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1. INTRODUCTION

In an hour's talk such as this it is impossible to cover all that is known on the subject, especially since in some respects detail is of the essence in dealing with exact solutions. The three major recent reviews of rather general character [1, 2, 3] have a total of over 1200 pages and that is before moving to more specialized reviews such as [4, 5]. (The short summary of what is new in [1] as compared with the first edition, given below as an appendix, illustrates some ways in which the field has changed in the last 25 years.) Thus I shall be selective, dwelling at some length on particular solutions rather than trying to cram in as many as possible in the time. For example, I give particular attention to the Schwarzschild solution, the first solution known that did not have constant curvature. I apologize to the authors of the many excellent pieces of work I do not touch on.

Before learning the techniques for finding solutions of the Einstein equations and discovering their properties, one should ask "why this is a worthwhile endeavour?". Some colleagues do seem to regard it as something of a backwater in the theory. The reason it is still important stems from the nonlinearity of general relativity, one of its essential features. To understand the meaning of the theory, there are really three approaches. One can seek to prove global results, such as are described in [6] and were the subject of the Isaac Newton Institute programme in progress at the time of writing (see `http://www.newton.cam.ac.uk/webseminars/pg+ws/2005/gmr/`). One can try to use approximation, either in the form of iterated perturbation methods or numerical solutions. Finally, one can use exact solutions: as Mason and Woodhouse said, "they combine tractability with nonlinearity, so they make it possible to explore nonlinear phenomena while working with explicit solutions" [7], and as we shall see below, they have had considerable impact on the theory.

We also need to ask: "what is a solution?". We could assume a form for the metric, list of the most commonly-rediscovered solutions:

1. Flat space
2. The Schwarzschild solution
3. The Kasner solutions
4. Plane waves
5. Conformally flat perfect fluids
6. The Taub-NUT family of solutions
7. Static spherically symmetric perfect fluids
8. Cylindrically symmetric stationary electrovac solutions
9. Plane symmetric fluid and electrovac solutions
10. Spherically symmetric shearfree fluids

The 7th and subsequent places in this list are actually rather open to debate, for example the Harris Zund class are strong contenders. I will say something later about how to recognize known solutions.

I should add that in this review I am going to stick firmly to the number of dimensions that I know I exist in, i.e. 4, and avoid trespassing on the 5- and higher-dimensional work described by others. The development of exact solutions in such theories seems to me to be for the most part still at the stage of using only very simple metric forms.

2. THE PRE-EXISTING SOLUTIONS

Since the title of this meeting is 'A century of relativity', not the 90 of General Relativity, I am obliged to begin at the beginning, meaning the first solution, the spacetime of special relativity, Minkowski space:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2. \tag{1}$$

It is flat, empty and is usually taken to have $\mathbb{R}^4$ topology.

The role of this solution has been considerable, for example:

- It provides the prototype for asymptotic flatness (see Ehlers' contribution to this volume)
- It is ideal for 'cutting and pasting', which was a technique used to great effect while the concepts of causal structure were being developed[2]
- It was the setting for the first work on acceleration horizons (the Unruh effect)
- In suitable coordinates, it is the Milne universe which is often used as the extreme Robertson-Walker model with $k = -1$. Apart from the use of this example in astrophysical predictions, this second choice of slicing of flat space, together with the three foliations of de Sitter space, helps to illustrate the fact that the 'open', 'flat' or 'closed' nature of spacetimes can depend on the slicing.

[2] As a student I had the pleasure of watching part of this work as member of Sciama's research group in Cambridge.

3.2. The Schwarzschild solution

The original form of the Schwarzschild solution [12] used a radial coordinate r which had its origin at the horizon: for clarity I denote this r_0 below. Schwarzschild also gave (contrary to some statements in the literature) the form most often quoted now:

$$ds^2 = r^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) + A^{-1}dr^2 - A dt^2, \tag{3}$$

where $A = 1 - 2m/r$. Here I have renamed Schwarzschild's R as r. Schwarzschild regarded this as an auxiliary form because it did not fulfil the condition $|\det(g)| = 1$ which Einstein had imposed in the initial formulation of General Relativity, a condition of course later discarded.

This solution initiated a discussion (see e.g. [26, 27]) on the meaning of the surface $r = 2m$ and the absence of a solution clearly analogous to a point mass in Newtonian theory. There is of course no alternative point mass solution to consider due to the uniqueness of (3) as the spherically symmetric vacuum solution (Birkhoff's theorem).

There are still authors who argue that Schwarzschild's original $r_0 = 0$ should be regarded as a singularity representing a point mass. The horizon clearly is not a regular point since the area of surrounding spheres has a limit $4\pi(2m)^2$ (which is part of the reason $r = 2m$ is now understood as a sphere, not a point). The argument can be made at various levels of sophistication, the simplest being that the metric components are singular (similar things could be said of the axis of spherical polars, but nobody argues this is singular!). To counter such arguments, it helps to note first that the horizon $r = 2m$ ($r_0 = 0$) is not in the coordinate patch for (3). (The waters here have been somewhat muddied by Hilbert's arguments that these coordinates could be continued to the interior, in which he overlooked that at the horizon the three coordinates (t, θ, φ) do not parametrize a three-dimensional manifold, as becomes immediately apparent on passing to the Kruskal-Szekeres picture.) The whole Schwarzschild patch $r > 2m$ is isometric to a region of the Kruskal-Szekeres solution (region I in the conformal diagram given as Fig. 1),

$$ds^2 = r^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) - 32m^3 r^{-1} e^{-r/2m} du \, dv, \tag{4}$$

where r is defined implicitly by the equations giving u and v in terms of the previous t and r:

$$u = -(r/2m - 1)^{1/2} e^{r/4m} e^{-t/4m}, \quad v = (r/2m - 1)^{1/2} e^{r/4m} e^{t/4m}. \tag{5}$$

By inspection of this form of the solution we see that considering the bounding surface $r = 2m$ at a given time as a point amounts to topologically identifying all the points on a sphere. One can produce an analogous effect by cutting and pasting flat space (in r_0-like coordinates).

One thing to note in these arguments is the ambivalence with which we treat coordinates. Introductions to General Relativity always emphasize the covariance of the equations, but practical examples often implicitly communicate the importance of particular coordinates. The problems this causes are seen at their worst when refereeing weak papers on exact solutions, where authors often refuse to accept that their solution is not new on the grounds its coordinate form is different from the known ones.

ric) or near-extreme charged black holes, a microscopic basis in string theory has been given [31]. Yet again the Schwarzschild solution's generalizations are playing a major role in advancing our understanding.

To summarize, the Schwarzschild solution assisted the development of our understanding of the following:

- there is no "point mass" solution in relativity, since there is no point centre;
- The role of coordinates has to be properly understood;
- global concepts such as
  black holes,
  event horizons,
  apparent horizons,
  trapped surfaces and the singularity theorems,
  cosmic censorship and naked singularities;
- PPN expansions; and
- Hawking radiation, QFT in curved spaces and black hole entropy.

3.3. The Robertson-Walker metric form: FLRW solutions

These are the solutions with the metric

$$ds^2 = -dt^2 + a^2(t)[dr^2 + \Sigma^2(r,k)(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)]. \quad (6)$$

where

$$\Sigma(r,k) = \sin r,\ r \text{ or } \sinh r, \text{respectively, when } k = 1, 0 \text{ or } -1. \quad (7)$$

It would be impossible to overemphasize the importance of these solutions in cosmology. They are fundamental to the inferences from observation of the presence on the cosmological scale of dark matter and "dark energy" (above the densities required by dynamics of galaxy clusters). These matters were discussed by other speakers at the meeting so I will not give details here.

There are many specific solutions due to Einstein himself, de Sitter, Friedman, Eddington, Lemaître, and so on (see Ch. 14 of [1]). The pivotal role of the ones due to Friedman and Lemaître led to those authors' names being coupled with the names of Robertson and Walker in the short name FLRW. The role of Robertson and Walker was that they independently elucidated the geometrical basis of the metric form in the 1930s.

One may also note the wide use of the Lemaître-Tolman-Bondi models, spherically symmetric inhomogeneous dust models, to describe collapse and voids, primordial black holes, and other inhomogeneities in cosmology (see [2]). For example, it has been shown that the number distribution of galaxies can be modelled by a non-evolving population in an LTB spacetime rather than an evolving population in FLRW.

pause in describing methods for finding solutions and discuss the uses of three of these solutions or solution classes.

4.2. The Taub-NUT solution

The metric in this case can be given as

$$ds^2 = -U^{-1}\,d\tau^2 + (2\ell)^2 U\,(d\psi + \cos\theta\,d\phi)^2 \;+\; (\tau^2+\ell^2)\,(d\theta^2 + \sin^2\theta\,d\phi^2), \quad (10)$$

where U, which is positive in the Taub region and negative in the NUT region, is given by

$$U(\tau) = -1 + 2\,\frac{m\tau + \ell^2}{\tau^2+\ell^2}. \quad (11)$$

This played such a role that Misner described it as a "counterexample to almost anything" [43].

Papers by Misner [43] and Misner and Taub [44] established that the Taub and NUT regions can be joined, that the NUT region contains closed timelike lines and no sensible Cauchy surfaces, that there are two inequivalent maximal analytic extensions of the Taub region (or one non-Hausdorff manifold with both extensions), that Taub-NUT space is nonsingular in the sense of a curvature singularity, and that there are geodesics of finite affine parameter length.

To summarize, it has the following properties:

- the topology of group orbits changes at the horizon;
- there are closed timelike lines in the NUT region;
- the boundary of the Taub region has closed null geodesics;
- there is geodesic incompleteness at finite affine parameter without a curvature singularity; and
- there are inequivalent extensions, or a non-Hausdorff one.

The solutions also have applications and generalizations outside strict general relativity, e.g. in string theory or Euclidean quantum gravity.

The solution has thus had a great influence on studies of exact solutions and cosmological models which are spatially-homogeneous, and more generally on those which are hypersurface-homogeneous and self-similar (see e.g. the discussion in [45]), on cosmology in general, and on our understanding of global analysis and singularities in space-times.

4.3. *pp*-waves and plane waves

The plane waves were first found by Brinkmann [19] but their significance, showing among other things that gravitational waves were definitely not a coordinate effect, was not appreciated until the 1950s, in work of Bondi, Pirani and Robinson [46], Peres [47], the summary in [53]. Moreover, the Hamilton-Jacobi and Klein-Gordon equations are separable in this metric, which is related to the fact that it has a non-trivial Killing tensor. It exhibits the phenomenon of an ergosphere, a region outside the black hole horizon but within which any particle has to corotate around the hole. There is a relation to work on characterization of stationary axisymmetric spacetimes by multipole moments: the Kerr solution has very specific relations between its moments which do not appear to be found in physical bodies of rotating fluid, posing a question about possible sources or the process of approach to the Kerr solution as the eventual black hole outcome of a collapse.

Work on its mathematical properties, however, is likely to be exceeded by the many papers on its astrophysical implications. For example, the ergosphere classically allows the Penrose process, in which a part of a body dividing within the ergosphere can emerge with more energy than the original body entered with. The wave version of this is superradiance, where the scattered wave has more energy than the incident wave, and this phenomenon was one of the stimuli to the laws of black hole mechanics later explained by Hawking. It is also related to the Blandford-Znajek mechanism [54], in which a magnetic field threading the black hole can extract rotational energy from the hole.

The most important of all the astrophysical uses of the Kerr solution is probably as a basis for accretion disk physics, thought, for example, to be responsible for the X-ray emission of X-ray binaries in the sky, and used in explanations of larger objects such as jets in active galactic nuclei. Observations of astronomical accretion disks are now suggesting the objects at their centres really are Kerr black holes, as the only way to explain both their short periods and other orbital data [55, 56]. Chandrasekhar remarked[4] "In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations, discovered by the New Zealand mathematician Roy Kerr, provides an absolutely exact representation of untold numbers of massive black holes that populate the Universe..."

4.5. Finding more solutions: generating techniques

A third set of novel techniques appearing for the first time in papers in the 50s and early 60s (by Buchdahl, Ehlers, and Bonnor, for instance [57, 58, 59]) took rather longer to grow to maturity than the first two methods described in this section: this third topic is that of the generating techniques. Although they exist for metrics with one Killing vector, they are most used for stationary axisymmetric solutions, and the other classes with two commuting Killing vectors: cylindrical waves and colliding plane waves, boost-rotation symmetric spacetimes and cosmologies with two commuting spacelike Killing vectors. They work not only for vacuum, but for other forms of matter with characteristic propagation speed equal to the speed of light: massless scalar fields (or 'stiff fluid' in the case of a timelike gradient of the field), (massless) neutrinos, and electromagnetism.

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[4] This remark came to my attention in Bicak's review [3].

5. FINDING AND USING MORE SOLUTIONS: RECENT DEVELOPMENTS

I now return to the fundamental question raised earlier: how can we compare solutions? In particular how can we test for local isometric equivalence? This is called the equivalence problem, and belongs to the general class of recognition problems. It has largely been answered, in theory and in practice, using ideas due to Cartan, Brans, Karlhede and others, with practical implementation and development by Åman, and later myself and my group, although in a formal sense its final step is undecideable. I will briefly describe the procedure now: a full description would be a lecture in itself. For a review see Ch. 9 of [1].

The key point is that scalar polynomial invariants, i.e. polynomials in the Riemann tensor and its derivatives in which all indices are contracted over, are insufficient. Instead we need curvature invariants of a more general sort, the 'Cartan invariants'. These are, for example, given by the components of the Weyl tensor referred to a tetrad chosen using the principal null directions.

These enable local characterization of the metric. The basic idea is to use invariantly defined tetrads, like the one from the principal null directions, and take components of the Riemann tensor and its derivatives in this frame. Thus the method has links with the Petrov classification and tetrad methods which were introduced in the 1950s and 1960s. Counting functionally independent invariants gives the dimension of the symmetry group, thus linking to the group theoretic ideas of that period, and additional information available can give the group structure.

These characterization methods can be used to check if solutions found are really new, or to search among known solutions for examples with desired local properties. They could in principle be used to find solutions, as well as classify known ones, and first examples of this method have been developed by Bradley, Karlhede and Marklund.

Other uses of this approach, i.e. the direct use of invariants, have been in understanding the limits of families of solutions without needing trial and error for the appropriate coordinate transformations (for example, studying the limits of the Schwarzschild family as $m \rightarrow \infty$); proving (non) existence of matchings by characterizing the geometries of the proposed matching surfaces (e.g. in work of my student Daniel Cox [60]); and in providing a method to 'unravel' directional singularities (in work of my student John Taylor [61]). The classifying quantities might also be used to give a topology on the space of solutions, which could even be of interest in numerical relativity.

Summarizing, the main methods for finding solutions in current use are still those outlined in Section 4, but the understanding and classification of these solutions can now be done in an invariant manner which should enable better use of the solutions found and may also provide a fresh and more invariant way to find more.

APPENDIX

The second edition of the exact solutions book contains about 400 pages of new material, covering hundreds of new solutions and references. In its preparation the authors read about 4000 new papers (as well as the 3000 read for the first edition). So there is far more

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