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A New Class of Vacuum Solutions of the Einstein Field Equations.(*)

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1. - Introduction.

In this paper the general solution of Einstein's empty space field equations, $R_{\mu\nu} = 0$, is obtained for a space where the metric has the form

$$g_{\mu\nu} = \eta_{\mu\nu} + l_\mu l_\nu. \tag{1.1}$$

Here $\eta_{\mu\nu}$ is the metric of Minkowski space in co-ordinates which are Cartesian but not necessarily rectangular, *i.e.*, $\eta_{\mu\nu}$ are constants, with signature $+++-$, and l_μ is null:

$$g^{\mu\nu} l_\mu l_\nu = 0. \tag{1.2}$$

The reason for considering vacuum solutions of the form (1.1) is that the contravariant components of the metric are easily expressed in terms of the covariant components. In fact,

$$g^{\mu\nu} = \eta^{\mu\nu} - l^\mu l^\nu, \tag{1.3}$$

where

$$l^\mu = g^{\mu\nu} l_\nu = \eta^{\mu\nu} l_\nu, \tag{1.4}$$

and the determinant

$$(-g) = -\det(g_{\mu\nu}) = 1. \tag{1.5}$$

It follows that if l_μ is null with respect to one of the two metrics $g_{\mu\nu}$ and $\eta_{\mu\nu}$, then it is null with respect to the other, *i.e.*, eq. (1.2) implies

$$\eta^{\mu\nu} l_\mu l_\nu = 0, \tag{1.6}$$

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and vice versa.

The new vacuum solutions have the following properties:

a) They include as special cases the Schwarzschild solution and the exterior solution of a rotating body which was recently discovered by one of us [1].

b) All vacuum solutions of the form (1.1) are algebraically degenerate in the sense of the Petrov-Pirani classification, l_μ being a multiple Debever-Penrose vector and thus geodesic and shear-free.

c) All vacuum solutions of the form (1.1) admit a one parameter group of motions. The Killing vector K^μ is at the same time a Killing vector of the flat Minkowski metric $\eta_{\mu\nu}$. In fact, with respect to $\eta_{\mu\nu}$, the motion is just a translation along a direction which may be time-like, space-like, or null in the Minkowski space. There are thus three cases:

$$\eta_{\mu\nu} K^\mu K^\nu \begin{cases} < 0, & \text{(Case I)} \\ > 0, & \text{(Case II)} \\ = 0. & \text{(Case III)} \end{cases} \tag{1.7}$$

d) In each of the three cases, the general solution is determined by one arbitrary analytic function of one complex variable.

e) If $g_{\mu\nu}$ admits a Killing vector other than K^μ, it also must be a Killing vector with respect to $\eta_{\mu\nu}$.

f) There are at most two essentially different ways of representing a vacuum metric in the form (1.1). For case I, apart from the special case mentioned in *a*), the representation (1.1) is, in fact, unique, so that the Riemannian space $g_{\mu\nu}$ determines uniquely the null vector field l_μ and the Minkowski space $\eta_{\mu\nu}$. Similar statements hold for cases II and III.

Together with their graduate student, Mr. George Debney, the authors have examined solutions of the (non-vacuum) Einstein-Maxwell equations where the metric has the form (1.1). Most of the results mentioned above apply to this more general case. This work is continuing.[1]

In the following sections, the derivation is outlined of the most important of the above properties, *i.e.*, properties *a*) to *d*). The full details of all the proofs will be published elsewhere.[2]

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[1] The results of that work were published in the paper: G. C. Debney, R. P. Kerr and A. Schild, Solutions of the Einstein and Einstein–Maxwell equations, *J. Math. Phys.* **10**, 1842 (1969) [editor].

[2] According to R. Kerr (private communication), the publication announced here is the Debney – Kerr – Schild paper mentioned in footnote 1 [editor].

2. - Outline of derivation.

A simple direct calculation of the Christoffel symbols $\left\{ {\rho \atop \mu\nu} \right\}$ for the metric (1.1), (1.3) and of $\left\{ {\nu \atop \mu\nu} \right\}$, $\left\{ {\rho \atop \mu\nu} \right\} l^\nu$, $\left\{ {\rho \atop \mu\nu} \right\} l^\nu l^\mu{}_{,\rho}$(*), and substitution into the vacuum field equation

$$R_{\mu\nu} l^\mu l^\nu = 0, \tag{2.1}$$

yields

$$l_{\mu,\nu} l^\nu l^\mu{}_{,\rho} l^\rho = 0. \tag{2.2}$$

By differentiating eq. (1.6), we also have

$$l_{\mu,\nu} l^\nu l^\mu = 0. \tag{2.3}$$

Thus, in the Minkowski metric $\eta_{\mu\nu}$, the vector $l^\mu{}_{,\nu} l^\nu$ is null and orthogonal to the null vector l^μ. In a four dimensional space with the signature $3+1$ of space-time, a vector ν^μ orthogonal to a null vector l^μ is either space-like or else a multiple of l^μ. Therefore

$$l^\mu{}_{,\nu} l^\nu = \mu l^\mu. \tag{2.4}$$

Also, it is easily shown that

$$l^\mu{}_{;\nu} l^\nu = l^\mu{}_{,\nu} l^\nu = \mu l^\mu. \tag{2.5}$$

Thus the field eq. (2.1) implies that the null field l^μ is geodesic. We can therefore define a new null vector field $k^\mu = l^\mu/(2H)^{1/2}$, so that

$$k_\mu k^\mu = 0, \qquad k^\mu{}_{;\nu} k^\nu = 0, \tag{2.6}$$

and

$$\begin{cases} g_{\mu\nu} = \eta_{\mu\nu} + 2H k_\mu k_\nu, \\ g^{\mu\nu} = \eta^{\mu\nu} - 2H k^\mu k^\nu, \\ k^\mu = g^{\mu\nu} k_\nu = \eta^{\mu\nu} k_\nu. \end{cases} \tag{2.7}$$

Another simple calculation gives[3]

$$R^\sigma{}_{\rho\mu\nu} k^\rho k^\nu = -\ddot{H} k^\sigma k_\mu, \tag{2.8}$$

(*) A comma denotes partial differentiation, *e.g.*, $l^\mu{}_{,\rho} = \partial l^\mu / \partial x^\rho$, a semi-colon, *e.g.*, $l^\mu{}_{;\nu}$ denotes covariant differentiation with respect to the metric $g_{\mu\nu}$.

[3]The index at the first k on the left corrected from σ to ρ [editor].

where the dot denotes differentiation in the null direction k^μ:

$$\dot{H} = H_\mu k^\mu. \tag{2.9}$$

A Riemannian space-time is caled algebraically degenerate (or algebraically special), in the sense of the Petrov-Pirani classification [2], if and only if there exists a null vector field k^μ which obeys the equations

$$k_{[\alpha} C_{\sigma]\rho\mu\nu} k^\rho k^\nu = 0, \tag{2.10}$$

where $C_{\sigma\rho\mu\nu}$ is Weyl's conformal curvature tensor. In vacuum, where the field equations $R_{\mu\nu} = 0$ hold, the Weyl tensor and the Riemann tensor coincide, *i.e.*, $C_{\sigma\rho\mu\nu} = R_{\sigma\rho\mu\nu}$. Thus the condition (2.10) reduces to

$$k_{[\alpha} R_{\sigma]\rho\mu\nu} k^\rho k^\nu = 0. \tag{2.11}$$

By (2.8), this is clearly satisfied. Thus all vacuum solutions of the form (1.1), or equivalently (2.7), are algebraically degenerate, k_μ being a multiple Debever-Penrose vector.

The Goldberg-Sachs [3] theorem immediately implies that the null field k^μ is not only geodesic but also shear-free.

We now introduce at each point of space-time a quasiorthogonal tetrad of null vectors $e_a{}^\mu$, where Latin or tetrad suffixes range and sum over 1, 2, 3, 4, and serve to label the different vectors of the tetrad, while Greek suffixes are, as before tensor suffixes. The tetrad vectors $e_1{}^\mu$ and $e_2{}^\mu$ are complex conjugates,

$$e_2{}^\mu = \bar{e}_1{}^\mu, \tag{2.12}$$

$e_3{}^\mu$ and

$$e_4{}^\mu = k^\mu \tag{2.13}$$

are real, and they satisfy the quasiorthogonality relations

$$e_a{}^\mu e_{b\mu} = g_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = g^{ab}. \tag{2.14}$$

We define the Ricci rotation coefficients by

$$\Gamma_{abc} = -e_{a\mu;\nu} e_b{}^\mu e_c{}^\nu = -\Gamma_{bac}. \tag{2.15}$$

The geodesic and shear-free character of $e_4{}^\mu = k^\mu$ is expressed analytically by

$$\Gamma_{m44} = \Gamma_{411} = \Gamma_{422} = 0. \tag{2.16}$$

The rotation coefficients

$$\Gamma_{241} = z = \theta - i\omega, \quad \Gamma_{142} = \bar{z} = \theta + i\omega \tag{2.17}$$

are Sachs' complex expansion [cf. Goldberg and Sachs [3], eq. (1.8)], $\theta = \mathrm{Re}[z]$ being the expansion rate of the congruence of null geodesics which have k^μ as tangents, and $\omega = -\mathrm{Im}[z]$ being the rotation rate. Throughout the following we shall consider only the general case

$$z \neq 0. \tag{2.18}$$

The tetrad or *slash* derivative of an invariant is defined by

$$T_{/a} = T_{;\mu} e_a{}^\mu = T_{,\mu} e_a{}^\mu. \tag{2.19}$$

By (2.9) and (2.13)

$$T_{/4} = \dot{T}. \tag{2.20}$$

For the commutator of two successive slash derivatives we obtain

$$T_{/ab} - T_{/ba} = T_{/m}(\Gamma^m_{\ ab} - \Gamma^m_{\ ba}), \tag{2.21}$$

where

$$\Gamma^m_{\ ab} = g^{mn}\Gamma_{nab}. \tag{2.22}$$

The tetrad components of the Ricci tensor are

$$\begin{aligned} R_{bc} &= R_{\mu\nu} e_b{}^\mu e_c{}^\nu = \\ &= \Gamma^a_{\ ba/c} - \Gamma^a_{\ bc/a} + \Gamma^m_{\ ba}\Gamma^a_{\ mc} - \Gamma^m_{\ bc}\Gamma^a_{\ ma} + \Gamma^a_{\ bm}(\Gamma^m_{\ ca} - \Gamma^m_{\ ac}). \end{aligned} \tag{2.23}$$

3. - The field equations.

We choose null co-ordinates in Minkowski space, complex conjugate co-ordinates $x^1 = \bar{\zeta}$, $x^2 = \zeta$, and real co-ordinates $x^3 = v$, $x^4 = u$, such that

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.1}$$

and

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 2\mathrm{d}\zeta \mathrm{d}\bar{\zeta} + 2\mathrm{d}u\mathrm{d}v + 2H(k_\mu \mathrm{d}x^\mu)^2. \tag{3.2}$$

A general field of[4] real null vectors in Minkowski space [cf. eq. (1.6)] is given by(*)

$$k_\mu \mathrm{d}x^\mu = e_{4\mu}\mathrm{d}x^\mu = \mathrm{d}u + Y\mathrm{d}\bar{\zeta} + \bar{Y}\mathrm{d}\zeta - Y\bar{Y}\mathrm{d}v, \tag{3.3}$$

where Y is an arbitrary complex function of position. The tetrad is now completed as follows,

$$\begin{cases} e_{1\mu}\mathrm{d}x^\mu = \mathrm{d}\bar{\zeta} - \bar{Y}\mathrm{d}v, \\ e_{2\mu}\mathrm{d}x^\mu = \mathrm{d}\zeta - Y\mathrm{d}v, \\ e_{3\mu}\mathrm{d}x^\mu = \mathrm{d}v - Hk_\mu \mathrm{d}x^\mu, \end{cases} \tag{3.4}$$

and is easily seen to satisfy the quasiorthogonality relations (2.14).

The conditions (2.16) for k_μ to be geodesic and shear-free become

$$Y_{/2} = Y_{/4} = \bar{Y}_{/1} = \bar{Y}_{/4} = 0. \tag{3.5}$$

We then find that

$$\Gamma_{m14} = \Gamma_{m24} = \Gamma_{m34} = 0. \tag{3.6}$$

Geometrically, this means that the tetrad vectors (3.4) are propagated parallelly along each curve of the congruence of null geodesics which have k^μ as tangents. Sachs' complex expansion, eq. (2.17), becomes

$$z = Y_{/1}, \quad \bar{z} = \bar{Y}_{/2}. \tag{3.7}$$

From here on, numerical suffixes on the Ricci tensor will always refer to its tetrad components R_{bc} as given by eq. (2.23).

The geodesic and shear-free condition (3.5) ensures that the following five field equations are satisfied:

$$R_{11} = R_{22} = R_{44} = R_{14} = R_{24} = 0. \tag{3.8}$$

The field equation $R_{12} = 0$ gives

$$H = e^{3P}(z + \bar{z}), \quad P_{/4} = 0, \tag{3.9}$$

[4] "or" corrected to "of" by the editor.

(*) The special case $k_\mu \mathrm{d}x^\mu = \mathrm{d}v$ can be included by a limiting process, *e.g.*, by starting with $k_\mu/(1 + Y\bar{Y})$ and then letting $Y \to \infty$. The complex function Y is the ratio of the two components of the spinor which corresponds to the null vector k_μ.

where P, like H, is real. The field equation $R_{34} = 0$ is now automatically satisfied. The field equation $R_{31} = 0$ and its complex conjugate $R_{32} = 0$ give

$$P_{/1} = (z/\overline{z})\overline{Y}_{/3}, \quad P_{/2} = (\overline{z}/z)Y_{/3}. \tag{3.10}$$

Finally, the field equation $R_{33} = 0$ gives

$$P_{/3} = \left(\frac{1}{z} + \frac{1}{\overline{z}}\right) Y_{/3}\overline{Y}_{/3}. \tag{3.11}$$

It is well worth pointing out that the calculations giving these results are by no means simple. In each of the above equations, all previously mentioned field equations have been assumed to hold, and frequent use has been made of the commutator relation (2.21) for slash derivatives.

It is now easy to show that P, Y and $\overline{Y}$ satisfy the pair of partial differential equations

$$X_{/4} = 0, \quad X_{/3} - X_{/1}Y_{/3}/z - X_{/2}\overline{Y}_{/3}/\overline{z} = 0, \tag{3.12}$$

and that Y and $\overline{Y}$ are functionally independent because $z \neq 0$ has been assumed. It follows that P is a function of Y and $\overline{Y}$:

$$P = P(Y, \overline{Y}), \tag{3.13}$$

and that the field eqs. (3.9), (3.10), (3.11) simplify to

$$P_Y = \overline{Y}_{/3}/\overline{z}, \quad P_{\overline{Y}} = Y_{/3}/z. \tag{3.14}$$

Differentiating the first of these equations in the direction of the tetrad vector $e_1{}^\mu$, we obtain, again using commutator relations for the slash derivatives, $zP_{YY} = zP_Y{}^2$, or equivalently

$$(\exp[-P])_{YY} = 0. \tag{3.15}$$

Similarly $(\exp[-P])_{\overline{Y}\overline{Y}} = 0$, so that $\exp[-P]$ must be bilinear in Y and $\overline{Y}$:

$$\exp[-P] = b + \overline{a}\overline{Y} + aY + cY\overline{Y}, \tag{3.16}$$

b and c being arbitrary real constants, and a and $\overline{a}$ arbitrary complex conjugate constants.

It is now easy to show that the operator

$$K = K^\mu \frac{\partial}{\partial x^\mu} = b\frac{\partial}{\partial u} + a\frac{\partial}{\partial \overline{\zeta}} + \overline{a}\frac{\partial}{\partial \zeta} - c\frac{\partial}{\partial v} \tag{3.17}$$

satisfies

$$(3.18)\quad KY=0,\quad K\overline{Y}=0,\quad Kz=KY_{/1}=0,\quad K\overline{z}=K\overline{Y}_{/2}=0.$$

Thus, by eqs. (3.2), (3.3), (3.9) and (3.13), K^μ is a Killing vector of our curved space-time with metric $g_{\mu\nu}$. Also, since K^μ has constant components, it is at the same time a translational Killing vector of the flat Minkowski space with metric $\eta_{\mu\nu}$.

4. - The general solution.

We are still free to perform Lorentz transformations which preserve the form $\mathrm{d}s_0{}^2=2\mathrm{d}\zeta\mathrm{d}\overline{\zeta}+2\mathrm{d}u\mathrm{d}v$ of the Minkowski metric. By a suitable choice of such a Lorentz transformation, we can simplify our solutions as follows:

Case I: $\eta_{\mu\nu}K^\mu K^\nu<0,\quad a=\overline{a}=0,\quad b=c>0.$

Then $K^\mu=b(0,0,-1,1)$ points along the time axis of Minkowski space. Equations (3.5) and (3.18) give

$$(4.1)\qquad \frac{\partial Y}{\partial v}+Y\frac{\partial Y}{\partial \zeta}=0,\quad \frac{\partial Y}{\partial \overline{\zeta}}-Y\frac{\partial Y}{\partial u}=0,\quad \frac{\partial Y}{\partial u}-\frac{\partial Y}{\partial v}=0.$$

The general solution is

$$(4.2)\qquad F=0,$$

$$(4.3)\qquad F(Y,\overline{\zeta},\zeta,u+v)\equiv\Phi(Y)+[Y^2\overline{\zeta}-\zeta+(u+v)Y],$$

where Φ is an arbitrary analytic function of the complex variable Y.

It is now easy to calculate

$$(4.4)\qquad z=Y_{/1}=-(1+Y\overline{Y})/F_Y,$$

$$(4.5)\qquad \exp[3P]=\sqrt{2}m(1+Y\overline{Y})^{-3},$$

where F_Y is the partial derivative with respect to Y of the function (4.3), where $m=2^{-1/2}b^{-3}$ is an arbitrary real constant, and where Y is determined as a function of the co-ordinates by the equation $F=0$. The solution takes the final form

$$(4.6)\qquad \begin{aligned}\mathrm{d}s^2&=2\mathrm{d}\zeta\mathrm{d}\overline{\zeta}+2\mathrm{d}u\mathrm{d}v-\\&\quad-4\sqrt{2}m\,\mathrm{Re}\left[\frac{1}{F_Y}\right]\cdot\left[\frac{\mathrm{d}u+Y\mathrm{d}\overline{\zeta}+\overline{Y}\mathrm{d}\zeta-Y\overline{Y}\mathrm{d}v}{1+Y\overline{Y}}\right]^2.\end{aligned}$$

Case II: $\eta_{\mu\nu}K^\mu K^\nu > 0, \quad a = \bar{a} = 0, \quad b = -c > 0.$

Then $K_\mu = b(0,0,1,1)$ points along a space axis of Minkowski space.

Arguments similar to those above, give the solution in final form

$$\begin{aligned} ds^2 &= 2d\zeta d\bar{\zeta} + 2du dv + \\ &+ 4\sqrt{2}m \,\mathrm{Re}\left[\frac{1}{F_Y}\right] \cdot \left[\frac{du + Yd\bar{\zeta} + \bar{Y}d\zeta - Y\bar{Y}dv}{1 - Y\bar{Y}}\right]^2, \end{aligned} \tag{4.7}$$

where $m = 2^{-1/2}b^{-3}$ is an arbitrary real constant, Y is a complex variable, F_Y is the partial derivative with respect to Y of the function

$$F(Y, \bar{\zeta}, \zeta, u - v) \equiv \Phi(Y) - [Y^2\bar{\zeta} + \zeta + (u - v)Y], \tag{4.8}$$

where Φ is an arbitrary analytic function of Y, and where Y is determined as a function of the co-ordinates by the equation

$$F = 0, \quad \Phi(Y) = Y^2\bar{\zeta} + \zeta + (u - v)Y. \tag{4.9}$$

Case III: $\eta_{\mu\nu}K^\mu K^\nu = 0, \quad a = \bar{a} = c = 0, \quad b > 0, \quad K^\mu = b(0,0,0,1).$

With the same notation and explanations as above, we have the solution in final form

$$\begin{aligned} ds^2 &= 2d\zeta d\bar{\zeta} + 2du dv - \\ &- 4\sqrt{2}m \,\mathrm{Re}\left[\frac{1}{F_Y}\right] \cdot (du + Yd\bar{\zeta} + \bar{Y}d\zeta - Y\bar{Y}dv)^2, \end{aligned} \tag{4.10}$$

$$F(Y, \bar{\zeta}, \zeta, v) \equiv \Phi(Y) + \zeta - vY, \tag{4.11}$$

$$F = 0, \quad \Phi(Y) = vY - \zeta. \tag{4.12}$$

5. - Gravitational field of rotating body and Schwarzschild solutions.

We consider here a particular solution with a Killing vector which is time-like with respect to the Minkowski metric $\eta_{\mu\nu}$. Equation (4.2) can be solved easily and explicitly if Φ is a quadratic polynomial in Y. We have chosen the direction of the time axis of Minkowski space so as to simplify the general solution to the form (4.6). We are, however, still free to perform translations and rotations in the 3-space orthogonal to the time direction. By a suitable choice of such a translation and rotation, the function F, eq. (4.3), with an arbitrary quadratic polynomial

$\Phi(Y)$ can be reduced to the standard form(*)

$$\Phi(Y) = -\sqrt{2}iaY, \quad a \geq 0. \tag{5.1}$$

We introduce rectangular Cartesian co-ordinates x, y, z, t in Minkowski space which are related to the null co-ordinates as follows

$$\sqrt{2}\zeta = x + iy, \quad \sqrt{2}v = z - t, \quad \sqrt{2}u = z + t. \tag{5.2}$$

Then the function F of eq. (4.3) is given by

$$F = 2^{-1/2}\left[(x - iy)Y^2 + 2(z - ia)Y - (x + iy)\right]. \tag{5.3}$$

We define a real function ρ by

$$\frac{x^2 + y^2}{\rho^2 + a^2} + \frac{z^2}{\rho^2} = 1. \tag{5.4}$$

The solution of $F = 0$ is

$$Y = \frac{(\rho - z)(\rho + ia)}{\rho(x - iy)}, \tag{5.5}$$

and for this Y we have

$$F_Y = \sqrt{2}\left[(x - iy)Y + (z - ia)\right] = \sqrt{2}(\rho^2 - iaz)/\rho, \tag{5.6}$$

$$Y\overline{Y} = \frac{\rho - z}{\rho + z}. \tag{5.7}$$

Substituting into eq. (4.6), we obtain

$$\begin{aligned} (5.8) \quad ds^2 &= dx^2 + dy^2 + dz^2 - dt^2 + \\ &+ \frac{2m\rho^3}{\rho^4 + a^2z^2}\left[dt + \frac{z}{\rho}dz + \frac{\rho}{\rho^2 + a^2}(xdx + ydy) + \frac{a}{\rho^2 + a^2}(xdy - ydx)\right]^2. \end{aligned}$$

This is the exterior gravitational field of a rotating body [1]. When we put $a = 0$, so that, by eq. (5.4), $\rho = r = (x^2 + y^2 + z^2)^{1/2}$, this reduces to the Schwarzschild metric

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m}{r}(dr + dt)^2 \tag{5.9}$$

in a form first given by Eddington [4].

(*) In this section we have changed notation slightly: a is now a real number which has nothing to do with the complex a of eq. (3.16); z is now one of the rectangular Cartesian co-ordinates x, y, z, t in Minkowski space, and not Sachs' complex expansion, eq. (2.17).

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INTERVENTI E DISCUSSIONI

– W. D. BEIGLBÖCK:

You gave us two solutions in your class, that have a fairly simple physical interpretation – namely the Schwarzschild and the Kerr solutions. Do you know whether there are any further solutions that would allow a physical interpretation?

– A. SCHILD:

The answer is «no». When $\Phi(Y)$ is not a quadratic polynomial the solutions are complicated, and we have not examined them.[9]

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[5] The full reference for the second paper is: *Quasi-stellar sources and graviational collapse.* Edited by I. Robinson, A. Schild, and E.L. Schucking, Chicago, University Press, 1965, p. 99 [editor].

[6] The correct full reference to the Petrov paper is: A. Z. Petrov, Klassifikacya prostranstv opredelyaushchikh polya tyagoteniya [The classification of spaces defining gravitational fields]. *Uchenye Zapiski Kazanskogo Gosudarstvennogo Universiteta im. V. I. Ulyanovicha-Lenina* **114**(8), 55 (1954). English translation (in the Golden Oldies series): *Gen. Rel. Grav.* **32**, 1665 (2000) [editor].

[7] The Goldberg – Sachs paper has been reprinted as a Golden Oldie, *Gen. Rel. Grav.* **41**, 433 (2009) [editor].

[8] The full reference to the first Robinson–Schild paper is: I. Robinson and A. Schild, Degeneracy and shear. In: *Proceedings on theory of gravitation. Conference in Warszawa and Jablonna, 25–31 July 1962.* Edited by L. Infeld. Gauthier–Villars, Paris and Państwowe Wydawnictwo Naukowe, Warszawa 1964, pp. 340–341 [editor].

[9] More solutions are known today. Information on them can be found, for example, in the following sources:

– A. Komar:

Ehlers, in his Royaumont paper[10], presented a general procedure for obtaining a Ricci flat stationary space from a given Ricci flat static space. What is the relationship (if any) between his procedure and yours?

– J. Ehlers:

The Schild-Kerr procedure is a definite generalization. The earlier procedure gives only one parametric sub-family of the Kerr-metrios.

– A. Schild:

Answer by Ehlers.

– J. Ehlers:

If my method of constructing stationary vacuum fields from static ones (described in my Royaumont lecture) is applied to the Schwarzschild solution a one-parameter (sub-) family of the Kerr-solution is obtained.

2) In connection with the problem of filling matter into the Kerr-solution I should like to point out: If the matter is to be a perfect fluid,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \quad (u_\lambda u^\lambda = 1),$$

and if the interior metric is to be stationary (like the exterior one) with Killing vector

$$\xi^\lambda = e^U u^\lambda$$

and if, moreover, g_{ab} is supposed to satisfy the Lichnerowicz junction conditions along the (hyper-) surface S of the body with ξ^λ of class

a) J. Bičak, in *Einstein's field equations and their physical implications. Selected essays in honour of Jürgen Ehlers.* Lecture Notes in Physics, vol. 540. Edited by B. Schmidt. Springer, Berlin 2000, p. 1.

b) N. Straumann, *General relativity.* Springer, Berlin 2004; section 7.2 (complete derivation via Ernst's equation).

c) H. A. Buchdahl, *17 Simple Lectures on General Relativity Theory,* Wiley, New York (N. Y.) 1981 (Lecture 13, p. 114).

d) H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations.* 2nd Edition. Cambridge University Press 2003. (Chapter 32, pp. 485 – 505: discussion of the Kerr–Schild metrics, Chapters 19 to 21, pp. 292 – 340: description of the stationary-axisymmetric solutions).

e) S. W. Hawking and G. F. R. Ellis, *The Large-scale Structure of Spaceetime.* Cambridge University Press 1973 [editor].

[10] The reference to this paper is: J. Ehlers, Transformations of static solutions of Einstein's gravitational field equations into different solutions by means of conformal mappings. In: *Les theories relativistes de la gravitation, Royaumont 1959,* edition CNRS, Paris 1965, pp. 275-283 [editor].

C^1, then S is given by U = cost. In fact, it follows from the above assumption that

$$U_{\lambda;\mu}u^{\mu} = -U_{,\lambda}, \quad p_{,\lambda}u^{\lambda} = 0,$$

consequently because of «Euler's» equation

$$(\rho + p)u_{\lambda;\mu}u^{\mu} = (\delta^{\mu}_{\lambda} - u^{\mu}u_{\lambda})p_{,\mu}$$

we obtain $(\rho + p)\mathrm{d}u = -\mathrm{d}p$ which implies, since $p = 0$ along S, that u = cost along S also. In consequence of this fact it is possible to study the «shape» of the field-producing body by means of data of the exterior field alone, at least under the hydrostatic circumstances stated above. In this way one may be able to get further support of the assumption that this exterior solution actually corresponds to a body in a state of stationary rotation.[11]

– A. SCHILD:

This is a very interesting remark. We shall look at the surfaces on which the stationary Killing vector has constant magnitude. Since the Kerr metric is known explicitly, this should be a straight forward problem.

[Alla discussione parteciparono anche G. MC VITTE, E. T. NEWMAN, C. MØLLER, H. J. TREDER, E. DEBEVER.]

– A. SCHILD (*additional remark*):

Professor Misner in a letter, and Professors Debever and Lichnerowicz at this meeting, kindly drew my attention to papers by Vaidya (*Nature* (1953))[12] and J. Hély (*Compt. Rend.* (1960))[13] which examine some space-times with electromagnetic radiation where the metric has the form of our eq. (1.1).

[11] No perfect fluid source for the Kerr solution has been found until today [editor].

[12] The full reference consists of three papers by P. C. Vaidya: The external field of a radiating star in general relativity, *Current Science* **12**, 183 (1943); The gravitational field of a radiating star, *Proceedings of the Indian Academy of Sciences* **A33**, 264 (1951); and "Newtonian" time in general relativity, *Nature* **171**, no 4345, 260 (1953). All three papers were reprinted in the Golden Oldies series in *Gen. Rel. Grav.* **31**, no 1, 137 (1999) [editor].

[13] The full reference is: J. Hély, *C. R Acad. Sci. Paris* **251**, pages 1981 and 2300 (1960) [editor].