# GRAVITATIONAL FIELD OF A SPINNING MASS AS AN EXAMPLE OF ALGEBRAICALLY SPECIAL METRICS

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Goldberg and Sachs[1] have proved that the algebraically special solutions of Einstein's empty-space field equations are characterized by the existence of a geodesic and shear-free ray congruence, $k_\mu$. Among these spaces are the plane-fronted waves and the Robinson-Trautman metrics[2] for which the congruence has nonvanishing divergence, but is hypersurface orthogonal.

In this note we shall present the class of solutions for which the congruence is diverging, and is not necessarily hypersurface orthogonal. The only previously known example of the general case is the Newman, Unti, and Tamburino metrics,[3] which is of Petrov Type D, and possesses a four-dimensional group of isometries.

If we introduce a complex null tetrad ($t^*$ is the complex conjugate of $t$), with

$$ds^2 = 2tt^* + 2mk,$$

then the coordinate system may be chosen so that

$$t = P(r + i\Delta)d\zeta,$$

$$k = du + 2\,\mathrm{Re}(\Omega d\zeta),$$

$$m = dr - 2\,\mathrm{Re}\{[(r - i\Delta)\dot{\Omega} + iD\Delta]d\zeta\} + \Big\{r\dot{P}/P + \mathrm{Re}[P^{-2}D(D^*\ln P + \dot{\Omega}^*)] + \frac{m_1 r - m_2\Delta}{r^2 + \Delta^2}\Big\}k; \quad (1)$$

where $\zeta$ is a complex coordinate, a dot denotes differentiation with respect to $u$, and the operator $D$ is defined by

$$D = \partial/\partial\zeta - \Omega\partial/\partial u.$$

$P$ is real, whereas $\Omega$ and $m$ (which is defined to be $m_1 + im_2$) are complex. They are all independent of the coordinate $r$. $\Delta$ is defined by

$$\Delta = \mathrm{Im}(P^{-2}D^*\Omega).$$

There are two natural choices that can be made for the coordinate system. Either (A) $P$ can be chosen to be unity, in which case $\Omega$ is complex, or (B) $\Omega$ can be taken pure imaginary, with $P$ different from unity. In case (A), the field equations are

$$(m - D^*D^*D\Omega) = |\partial_u D\Omega|^2, \quad (2)$$

$$\mathrm{Im}(m - D^*D^*D\Omega) = 0, \quad (3)$$

$$D^*m = 3m\dot{\Omega}. \quad (4)$$

The second coordinate system is probably better, but it gives more complicated field equations.

It will be observed that if $m$ is zero then the field equations are integrable. These spaces correspond to the Type-III and null spaces with nonzero divergence. If $m \neq 0$, then there are certain integrability conditions which must be satisfied by Eqs. (2)-(4). These may be solved for $m$ as a function of $\Omega$ and its derivatives provided that either $\dot{\Delta}$ or $\ddot{\Omega}$ is nonzero. This expression for $m$ may then be substituted back into the field equations giving conditions on $\Omega$ and its derivatives, from which further integrability conditions are extracted.

If both $\dot{\Delta}$ and $\ddot{\Omega}$ are zero, then we may transform the metric to a coordinate system in which $\Omega$ is pure imaginary and $P \not\equiv 1$, with $\dot{\Omega} = \dot{P} = 0$. The field equations then become

$$m = cu + A + iB,$$

where $c$ is a real constant, and

$$P^{-2}\nabla[P^{-2}\nabla(\ln P)] = 2c, \quad \nabla = \partial^2/\partial\zeta\partial\zeta^*.$$

$A$, $B$, and $\Omega$, which are all independent of $u$ and $r$, are determined by

$$iB = \tfrac{1}{2}P^{-2}\nabla(P^{-2}\partial\Omega/\partial\zeta) - P^{-4}(\partial\Omega/\partial\zeta)\nabla(\ln P),$$

$$\nabla B = ic\partial\Omega/\partial\zeta,$$

$$(\partial/\partial\zeta)(A - iB) = c\Omega,$$

where $\zeta = \xi + i\eta$. If $c$ is zero, then $\partial/\partial u$ is a Killing vector.

Among the solutions of these equations, there is one which is stationary ($c = 0$) and also is axially symmetric. Like the Schwarzschild metric, which it contains, it is Type D. Also, $B$ is zero, and $m$ is a real constant, the Schwarzschild mass. The metric is

$$ds^2 = (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2) + 2(du + a\sin^2\theta d\phi) \times (dr + a\sin^2\theta d\phi) - \left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right) \times (du + a\sin^2\theta d\phi)^2,$$

where $a$ is a real constant. This may be transformed to an asymptotically flat coordinate system by the transformation

$$(r - ia)e^{i\phi}\sin\theta = x + iy, \quad r\cos\theta = z, \quad u = t + r,$$

the metric becoming

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2mr^3}{r^4 + a^2z^2}(k)^2,$$

$$(r^2 + a^2)rk = r^2(xdx + ydy) + ar(xdy - ydx) + (r^2 + a^2)(zdz + rdt). \qquad (5)$$

This function $r$ is defined by

$$r^4 - (R^2 - a^2)r^2 - a^2z^2 = 0, \quad R^2 = x^2 + y^2 + z^2,$$

so that asymptotically $r = R + O(R^{-1})$. In this coordinate system the solution is analytic everywhere, except at $R = a$, $z = 0$.

If we expand the metric in Eq. (5) as a power series in $m$ and $a$, assuming $m$ to be of order two and $a$ of order one, and compare it with the third-order Einstein-Infeld-Hoffmann approximation for a spinning particle, we find that $m$ is the Schwarzschild mass and $ma$ the angular momentum about the $z$ axis. It has no higher order multipole moments in this approximation. Since there is no invariant definition of the moments in the exact theory, one cannot say what they are, except that they are small. It would be desirable to calculate an interior solution to get more insight into this.

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[2]I. Robinson and A. Trautman, Proc. Roy. Soc. (London) A265, 463 (1961).

[3]E. Newman, L. Tamburino, and T. Unti (to be published).