UNIFORMITY OF THE MAGNETIC FIELD IN A HELMHOLTZ COIL CONFIGURATION

1. On-Axis Case

The magnitude of the magnetic field $|B|$ a distance x from the center of a single coil of radius a with N turns and carrying a current I is (the derivation is given below for the general off-axis case)

$$|B| = \frac{\mu_0 N I a^2}{2\left(x^2 + a^2\right)^{3/2}}, \tag{1}$$

and for the current direction shown in Figure 1 is directed toward the negative x direction (right hand rule).

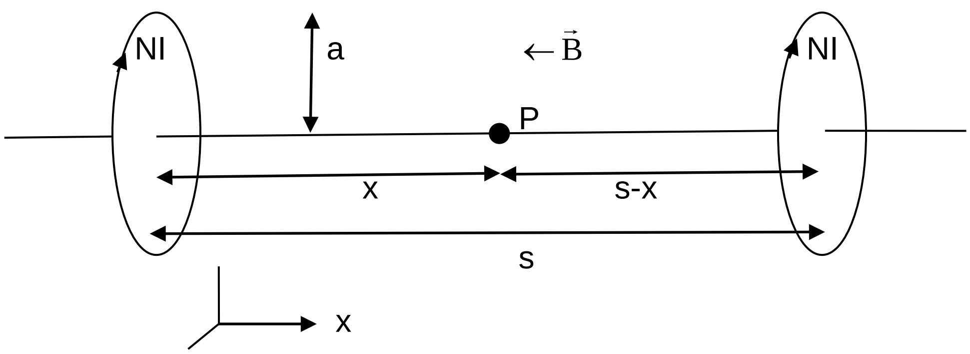

Figure 1

Define the separation of the two Helmholtz coils to be s and the distance of an on-axis point P from each of the coils to be x and $(s-x)$ (see Figure 1). The combined field at point P is then

$$|B| = \frac{\mu_0 N I a^2}{2}\left\{\frac{1}{\left(x^2+a^2\right)^{3/2}} + \frac{1}{\left[(s-x)^2+a^2\right]^{3/2}}\right\}. \tag{2}$$

The condition for $|B|$ to vary least with position along the x-axis is that $\left(d|B|/dx\right)$ be a minimum so that $\left(d^2|B|/dx^2\right)=0$:

$$\frac{d|B|}{dx} = B_0\left\{-\frac{3}{2}\left(x^2+a^2\right)^{-5/2}(2x) - \frac{3}{2}\left[(s-x)^2+a^2\right]^{-5/2}\left[2(s-x)\right](-1)\right\}$$
$$= 3B_0\left\{-x\left(x^2+a^2\right)^{-5/2} + (s-x)\left[(s-x)^2+a^2\right]^{-5/2}\right\} \tag{3}$$

where

$$B_0 \equiv \frac{\mu_0 N I a^2}{2}. \tag{4}$$

Then

$$\frac{d^2|B|}{dx^2} = 3B_0\left\{\begin{array}{l} -\left(x^2+a^2\right)^{-5/2} - x\left(\frac{-5}{2}\right)\left(x^2+a^2\right)^{-7/2}(2x) \\ -\left[(s-x)^2+a^2\right]^{-5/2} - \left(\frac{5}{2}\right)(s-x)\left[(s-x)^2+a^2\right]^{-7/2}\left[2(s-x)\right](-1) \end{array}\right\}$$

$$= 3B_0\left\{\begin{array}{l} -\left(x^2+a^2\right)^{-5/2} + 5x^2\left(x^2+a^2\right)^{-7/2} \\ -\left[(s-x)^2+a^2\right]^{-5/2} 5(s-x)^2\left[(s-x)^2+a^2\right]^{-7/2} \end{array}\right\}$$
$$= 0. \tag{5}$$

Insertion of the condition $x = s/2$ into eq. (5) yields

$$-\left(\frac{s^2}{4}+a^2\right)^{-5/2} + 5\left(\frac{s^2}{4}\right)\left(\frac{s^2}{4}+a^2\right)^{-7/2} - \left(\frac{s^2}{4}+a^2\right)^{-5/2} + 5\left(\frac{s^2}{4}\right)\left(\frac{s^2}{4}+a^2\right)^{-7/2} = 0$$

$$\Rightarrow -\left(\frac{s^2}{4}+a^2\right)^{-5/2} + 5\left(\frac{s^2}{4}\right)\left(\frac{s^2}{4}+a^2\right)^{-7/2} = 0, \tag{6}$$

and multiplying through by $\left(\frac{s^2}{4}+a^2\right)^{7/2} \neq 0$ gives the desired answer:

$$-\left(\frac{s^2}{4}+a^2\right)+5\left(\frac{s^2}{4}\right)=0 \Rightarrow s=\pm a\,. \qquad (7)$$

THUS THE OPTIMUM SEPARATION s BETWEEN THE COILS IS THE COIL RADIUS a.

The minimum value of $\left(\frac{d|B|}{dx}\right)$ is obtained by inserting $s=a$ and $x=s/2$ into eq. (3):

$$\left.\frac{d|B|}{dx}\right|_{\min} = 3B_0\left\{-\frac{a}{2}\left(\frac{5a^2}{2}\right)^{-5/2}+\frac{a}{2}\left(\frac{5a^2}{2}\right)^{-5/2}\right\}=0\,. \qquad (8)$$

The values of $|B|$ at the center of each coil and half way between the coils are

$$x=0: \qquad |B|=\frac{B_0}{a^3}\left(1+\frac{1}{2^{3/2}}\right)=\frac{(1.3536)B_0}{a^3} \qquad (9)$$

and

$$x=a/2: \qquad |B|=\frac{B_0}{a^3}\left[\frac{2}{(5/4)^{3/2}}\right]=\frac{(1.4311)B_0}{a^3}\,, \qquad (10)$$

yielding a ratio of $(1.3536/1.4311)=0.946$. Thus the on–axis magnetic field between the coils is constant to within about 5.4%.

A GNU Octave plot of $|B|$ vs. x/a computed from equation (2) is included in Figure 3 below as a special case $(h=0)$ for positions a distance h off-axis. The on-axis value of $|B|$ varies by less than 1% for $0.8 \geq x/a \geq 0.2$ and by less than 0.1% for $0.6 \geq x/a \geq 0.4$.

2. Off-Axis Case

Use the Biot–Savart law:

$$\left|d\vec{B}\right| = \left(\frac{\mu_0 NI}{4\pi\left|r\right|^2}\right) d\vec{s} \times d\vec{r} \tag{11}$$

where $\left|\mathrm{d}s\right| = a\,\mathrm{d}\theta$. Consider a point that is an orthogonal distance h from the central axis that passes through the center of each coil (see Figure 2), and let a, NI, and x be the same as above.

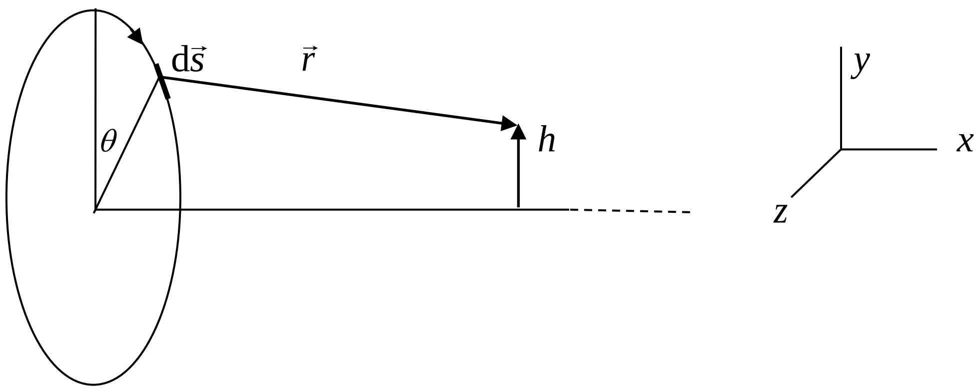

Figure 2

The components of $\mathrm{d}\vec{s}$ are

$$\left.\begin{aligned} ds_x &= 0 \\ ds_y &= -ds\sin\theta = -a\sin\theta d\theta \\ ds_z &= -ds\cos\theta = -a\cos\theta d\theta \end{aligned}\right\} \tag{12}$$

and the components of $\vec{r}$ are

$$\left.\begin{aligned} r_x &= x \\ r_y &= h - a\cos\theta \\ r_z &= +a\sin\theta \end{aligned}\right\} \tag{13}$$

so that

$$\left.\begin{aligned}|r| &= \left[x^2 + a^2\sin^2\theta + (h - a\cos\theta)^2\right]^{1/2} \\ &= \left[x^2 + a^2 + h^2 - 2ah\cos\theta\right]^{1/2}\end{aligned}\right\}. \tag{14}$$

Thus

$$\left.\begin{aligned} d\vec{s}\times\vec{r} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k} \\ 0 & -a\sin\theta d\theta & -a\cos\theta d\theta \\ x & h - a\cos\theta & a\sin\theta\end{vmatrix} \\ &= \hat{i}\left[-a^2\sin^2\theta d\theta - a\cos\theta(h - a\cos\theta)\right] - \hat{j}\left[ax\cos\theta d\theta\right] + \hat{k}\left[ax\sin\theta d\theta\right]\end{aligned}\right\} \tag{15}$$

so that, with $B_0 = \mu_0 NIa^2/2$ as before, the components of the field from the leftmost coil in Fig.1 are

$$\left.\begin{aligned} B_x &= B_0\int_0^{2\pi}\frac{-a^2 d\theta}{\left[x^2 + a^2 + h^2 - 2ah\cos\theta\right]^{3/2}} + \int_0^{2\pi}\frac{ah\cos\theta d\theta}{\left[x^2 + a^2 + h^2 - 2ah\cos\theta\right]^{3/2}}; \\ B_y &= B_0\int_0^{2\pi}\frac{ax\cos\theta d\theta}{\left[x^2 + a^2 + h^2 - 2ah\cos\theta\right]^{3/2}}; \\ B_z &= B_0\int_0^{2\pi}\frac{ax\sin\theta d\theta}{\left[x^2 + a^2 + h^2 - 2ah\cos\theta\right]^{3/2}}.\end{aligned}\right\} \tag{16}$$

For the rightmost coil

$$\left.\begin{aligned}
B_x &= B_0 \int_0^{2\pi} \frac{-a^2 d\theta}{\left[(s-x)^2 + a^2 + h^2 - 2ah\cos\theta\right]^{3/2}} + \int_0^{2\pi} \frac{ah\cos\theta d\theta}{\left[(s-x)^2 + a^2 + h^2 - 2ah\cos\theta\right]^{3/2}};\\
B_y &= B_0 \int_0^{2\pi} \frac{a(s-x)\cos\theta d\theta}{\left[(s-x)^2 + a^2 + h^2 - 2ah\cos\theta\right]^{3/2}};\\
B_z &= B_0 \int_0^{2\pi} \frac{a(s-x)\sin\theta d\theta}{\left[(s-x)^2 + a^2 + h^2 - 2ah\cos\theta\right]^{3/2}}.
\end{aligned}\right\} \quad (17)$$

There are three integrals in eqs. (16) and (17) of which only one is simple:

$$B_z = B_0 \int_0^{2\pi} \frac{ax\sin\theta d\theta}{\left[x^2 + a^2 + h^2 - 2ah\cos\theta\right]^{3/2}} = B_0 \int_0^{2\pi} \frac{axd\cos\theta}{\left[x^2 + a^2 + h^2 - 2ah\cos\theta\right]^{3/2}} = 0. \quad (18)$$

Physical symmetry pleasantly implies eq (18) as well. The other integrals for $h = 0$ are

$$B_x = B_0 \int_0^{2\pi} \frac{-a^2 d\theta}{\left[x^2 + a^2\right]^{3/2}} = \frac{-a^2 B_0 2\pi}{\left[x^2 + a^2\right]^{3/2}}, \quad (19)$$

$$B_y = B_0 \int_0^{2\pi} \frac{ax\cos\theta d\theta}{\left[x^2 + a^2\right]^{3/2}} = \frac{B_0 ax}{\left[x^2 + a^2\right]^{3/2}} \int_0^{2\pi} \cos\theta d\theta = 0. \quad (20)$$

For $h = 0$ eqs. (4) and (19) yield eq. (1). For $h \neq 0$ the integrals for B_x and B_y in eqs. (16) and (17) are elliptical but are easily computed numerically. Plots of four magnetic field properties vs. distance x from the leftmost coil are shown in Figure 3 below for four values of $h/a = 0;\ 0.2;\ 0.4;\ 0.6.$ The four properties are

(1) Magnitude of the field $|B|$;

(2) x component of $B = B_x$;

(3) y component of $B = B_y$;

(4) Angle (in degrees) between B_y and B_x = $\text{atan}(|B_y|/|B_x|)$.

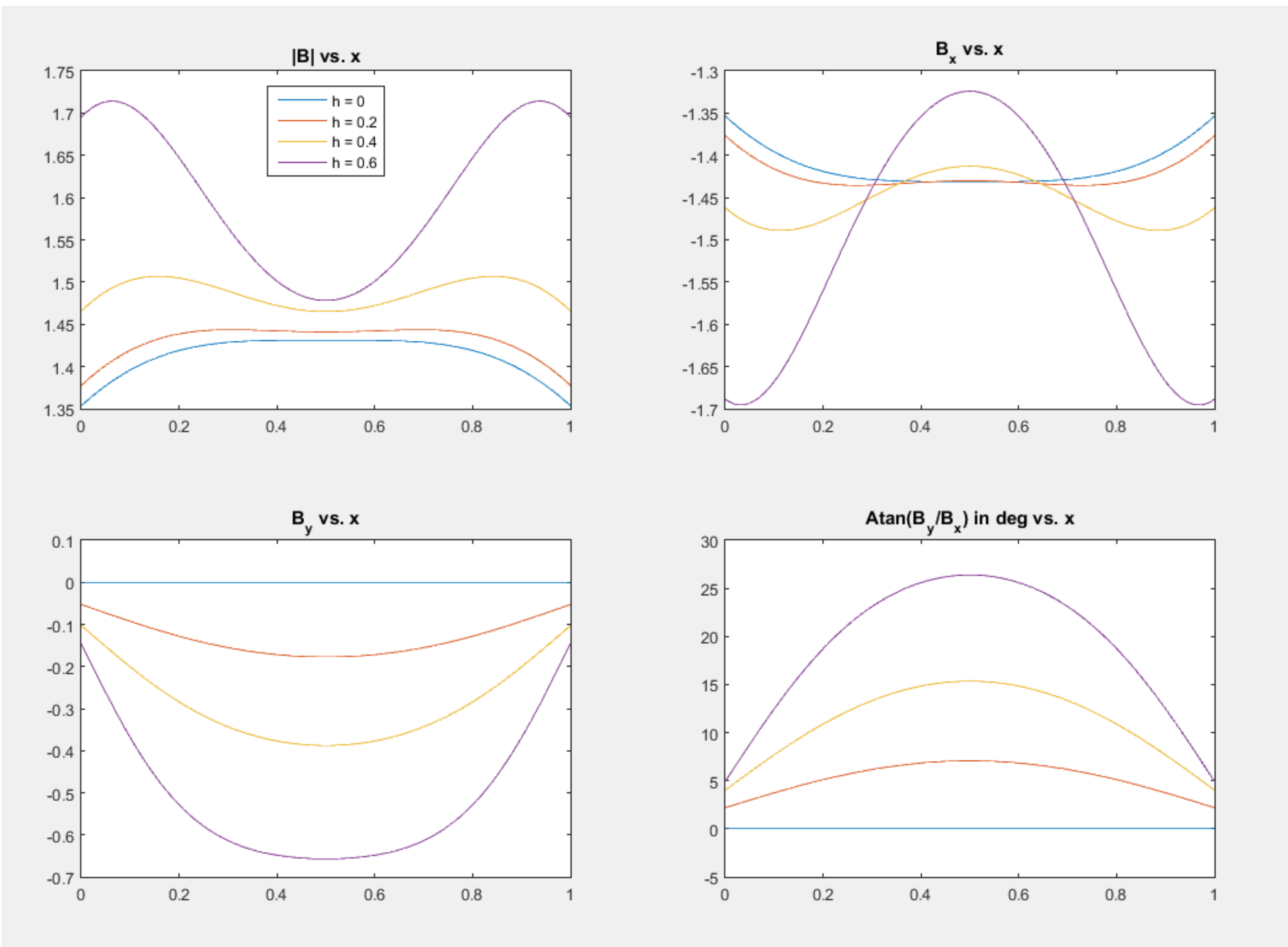

Figure 3