

## UNIFORMITY OF THE MAGNETIC FIELD IN A HELMHOLTZ COIL CONFIGURATION

### 1. On-Axis Case

The magnitude of the magnetic field  $|B|$  a distance  $x$  from the center of a single coil of radius  $a$  with  $N$  turns and carrying a current  $I$  is (the derivation is given below for the general off-axis case)

$$|B| = \frac{\mu_0 N I a^2}{2(x^2 + a^2)^{3/2}}, \quad (1)$$

and for the current direction shown in Figure 1 is directed toward the negative  $x$  direction (right hand rule).

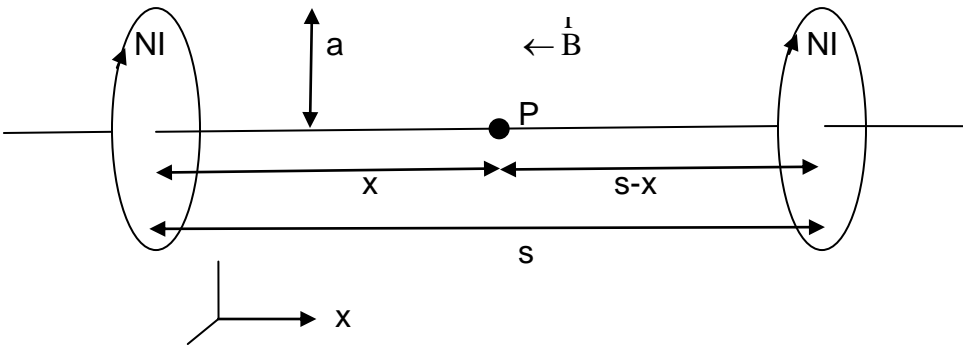


Figure 1

The planes of the coils are perpendicular to the axis. Define the separation of the two Helmholtz coils to be  $s$  and the distance of an on-axis point  $P$  from each of the coils to be  $x$  and  $(s - x)$  (see Figure 1). The combined field at point  $P$  is then

$$|B| = \frac{\mu_0 N I a^2}{2} \left\{ \frac{1}{(x^2 + a^2)^{3/2}} + \frac{1}{[(s-x)^2 + a^2]^{3/2}} \right\}. \quad (2)$$

The condition for  $|B|$  to vary least with position along the  $x$ -axis is that  $(d|B|/dx)$  be a minimum so that  $(d^2|B|/dx^2) = 0$ :

$$\begin{aligned}\frac{d|B|}{dx} &= B_0 \left\{ -\frac{3}{2}(x^2 + a^2)^{-5/2} (2x) - \frac{3}{2}[(s-x)^2 + a^2]^{-5/2} [2(s-x)](-1) \right\} \\ &= 3B_0 \left\{ -x(x^2 + a^2)^{-5/2} + (s-x)[(s-x)^2 + a^2]^{-5/2} \right\}\end{aligned}\quad (3)$$

where

$$B_0 \equiv \frac{\mu_0 N I a^2}{2} . \quad (4)$$

Then

$$\begin{aligned}\frac{d^2|B|}{dx^2} &= 3B_0 \left\{ \begin{aligned} &-(x^2 + a^2)^{-5/2} - x\left(\frac{-5}{2}\right)(x^2 + a^2)^{-7/2} (2x) \\ &-[(s-x)^2 + a^2]^{-5/2} - \left(\frac{5}{2}\right)(s-x)[(s-x)^2 + a^2]^{-7/2} [2(s-x)](-1) \end{aligned} \right\} \\ &= 3B_0 \left\{ \begin{aligned} &-(x^2 + a^2)^{-5/2} + 5x^2(x^2 + a^2)^{-7/2} \\ &-[(s-x)^2 + a^2]^{-5/2} + 5(s-x)^2[(s-x)^2 + a^2]^{-7/2} \end{aligned} \right\} \\ &= 0.\end{aligned}\quad (5)$$

Insertion of the condition  $x = s / 2$  into eq. (5) yields

$$\begin{aligned}-\left(\frac{s^2}{4} + a^2\right)^{-5/2} + 5\left(\frac{s^2}{4}\right)\left(\frac{s^2}{4} + a^2\right)^{-7/2} - \left(\frac{s^2}{4} + a^2\right)^{-5/2} + 5\left(\frac{s^2}{4}\right)\left(\frac{s^2}{4} + a^2\right)^{-7/2} &= 0 \\ \Rightarrow -\left(\frac{s^2}{4} + a^2\right)^{-5/2} + 5\left(\frac{s^2}{4}\right)\left(\frac{s^2}{4} + a^2\right)^{-7/2} &= 0,\end{aligned}\quad (6)$$

and multiplying through by  $\left(\frac{s^2}{4} + a^2\right)^{7/2} \neq 0$  gives the desired answer:

$$-\left(\frac{s^2}{4} + a^2\right) + 5\left(\frac{s^2}{4}\right) = 0 \Rightarrow s = \pm a. \quad (7)$$

THUS THE OPTIMUM SEPARATION  $s$  BETWEEN THE COILS IS THE COIL RADIUS  $a$ .

The minimum value of  $\left(\frac{d|B|}{dx}\right)$  is obtained by inserting  $s = a$  and  $x = s/2$  into eq. (3):

$$\left.\frac{d|B|}{dx}\right|_{\min} = 3B_0 \left\{ -\frac{a}{2} \left(\frac{5a^2}{2}\right)^{-5/2} + \frac{a}{2} \left(\frac{5a^2}{2}\right)^{-5/2} \right\} = 0. \quad (8)$$

The values of  $|B|$  at the center of each coil and half way between the coils are

$$x = 0: \quad |B| = \frac{B_0}{a^3} \left(1 + \frac{1}{2^{3/2}}\right) = \frac{(1.3536)B_0}{a^3} \quad (9)$$

and

$$x = a/2: \quad |B| = \frac{B_0}{a^3} \left[ \frac{2}{(5/4)^{3/2}} \right] = \frac{(1.4311)B_0}{a^3}, \quad (10)$$

yielding a ratio of  $(1.3536/1.4311) = 0.946$ . Thus the on-axis magnetic field between the coils is constant to within about 5.4%.

A GNU Octave plot of  $|B|$  vs.  $x/a$  computed from equation (2) is included in Figure 3 below as a special case ( $h=0$ ) for positions a distance  $h$  off-axis. The on-axis value of  $|B|$  varies by less than 1% for  $0.8 \geq x/a \geq 0.2$  and by less than 0.1% for  $0.6 \geq x/a \geq 0.4$ .

## 2. Off-Axis Case

Use the Biot–Savart law

$$|d\vec{B}| = \left( \frac{\mu_0 NI}{4\pi |r^2|} \right) d\vec{s} \times d\vec{r}, \quad (11)$$

where  $|ds| = a d\theta$ . Consider a point that is an orthogonal distance  $h$  from the central axis that passes through the center of each coil (see Figure 2), and let  $a$ ,  $NI$ , and  $x$  be the same as above.

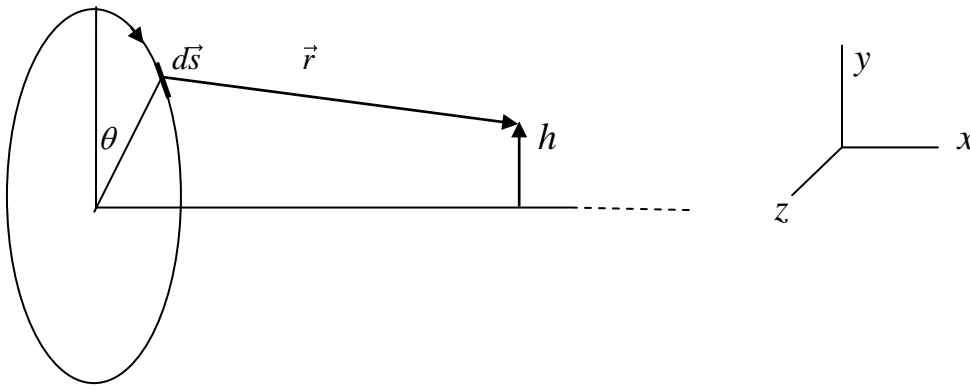


Figure 2

The components of  $d\vec{s}$  are

$$\left. \begin{aligned} ds_x &= 0 \\ ds_y &= -ds \sin \theta = -a \sin \theta d\theta \\ ds_z &= -ds \cos \theta = -a \cos \theta d\theta \end{aligned} \right\} \quad (12)$$

and the components of  $\vec{r}$  are

$$\left. \begin{aligned} r_x &= x \\ r_y &= h - a \cos \theta \\ r_z &= +a \sin \theta \end{aligned} \right\} \quad (13)$$

Thus

$$|r| = \left[ \begin{aligned} & x^2 + a^2 \sin^2 \theta + (h - a \cos \theta)^2 \\ & = [x^2 + a^2 + h^2 - 2ah \cos \theta]^{1/2} \end{aligned} \right]^{1/2}. \quad (14)$$

and

$$d\vec{s} \times \vec{r} = \left. \begin{aligned} & \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -a \sin \theta d\theta & -a \cos \theta d\theta \\ x & h - a \cos \theta & a \sin \theta \end{vmatrix} \\ & = \hat{i} [-a^2 \sin^2 \theta d\theta - a \cos \theta (h - a \cos \theta)] - \hat{j} [ax \cos \theta d\theta] + \hat{k} [ax \sin \theta d\theta] \end{aligned} \right\} \quad (15)$$

Thus, with  $B_0 = \mu_0 N I a^2 / 2$  as before, the components of the field from the leftmost coil in Fig.1 are

$$\left. \begin{aligned} B_x &= B_0 \left[ \int_0^{2\pi} \frac{-a^2 d\theta}{[x^2 + a^2 + h^2 - 2ah \cos \theta]^{3/2}} + \int_0^{2\pi} \frac{ah \cos \theta d\theta}{[x^2 + a^2 + h^2 - 2ah \cos \theta]^{3/2}} \right]; \\ B_y &= B_0 \int_0^{2\pi} \frac{ax \cos \theta d\theta}{[x^2 + a^2 + h^2 - 2ah \cos \theta]^{3/2}}; \\ B_z &= B_0 \int_0^{2\pi} \frac{ax \sin \theta d\theta}{[x^2 + a^2 + h^2 - 2ah \cos \theta]^{3/2}}. \end{aligned} \right\} \quad (15)$$

For the rightmost coil

$$\left. \begin{aligned}
B_x &= B_0 \int_0^{2\pi} \frac{-a^2 d\theta}{\left[ (s-x)^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}} + \int_0^{2\pi} \frac{ah \cos \theta d\theta}{\left[ (s-x)^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}}; \\
B_y &= B_0 \int_0^{2\pi} \frac{a(s-x) \cos \theta d\theta}{\left[ (s-x)^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}}; \\
B_z &= B_0 \int_0^{2\pi} \frac{a(s-x) \sin \theta d\theta}{\left[ (s-x)^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}}.
\end{aligned} \right\} \quad (16)$$

There are three integrals in eqs. (15) and (16) of which only one is simple:

$$B_z = B_0 \int_0^{2\pi} \frac{ax \sin \theta d\theta}{\left[ x^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}} = B_0 \int_0^{2\pi} \frac{ax d \cos \theta}{\left[ x^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}} = 0. \quad (17)$$

Physical symmetry happily implies eq (17) as well. The other integrals for  $h=0$  are

$$B_x = B_0 \int_0^{2\pi} \frac{-a^2 d\theta}{\left[ x^2 + a^2 \right]^{3/2}} = \frac{-a^2 B_0 2\pi}{\left[ x^2 + a^2 \right]^{3/2}}, \quad (18)$$

$$B_y = B_0 \int_0^{2\pi} \frac{ax \cos \theta d\theta}{\left[ x^2 + a^2 \right]^{3/2}} = \frac{B_0 ax}{\left[ x^2 + a^2 \right]^{3/2}} \int_0^{2\pi} \cos \theta d\theta = 0. \quad (19)$$

For  $h=0$  eqs. (4) and (18) yield eq. (1). For  $h \neq 0$  the integrals for  $B_x$  and  $B_y$  in eqs. (15) and (17) are elliptical but are easily computed numerically. Plots of four magnetic field properties vs. distance  $x$  from the leftmost coil are shown in Figure 3 below for four values of  $h/a=0; 0.2; 0.4; 0.6$ . The plotted properties are

- (1) Magnitude of the field  $|B|$ ;
- (2)  $x$  component of  $B = B_x$ ;
- (3)  $y$  component of  $B = B_y$ ;
- (4) Angle (in degrees) between  $B_y$  and  $B_x = \text{atan}(|B_y|/|B_x|)$ .

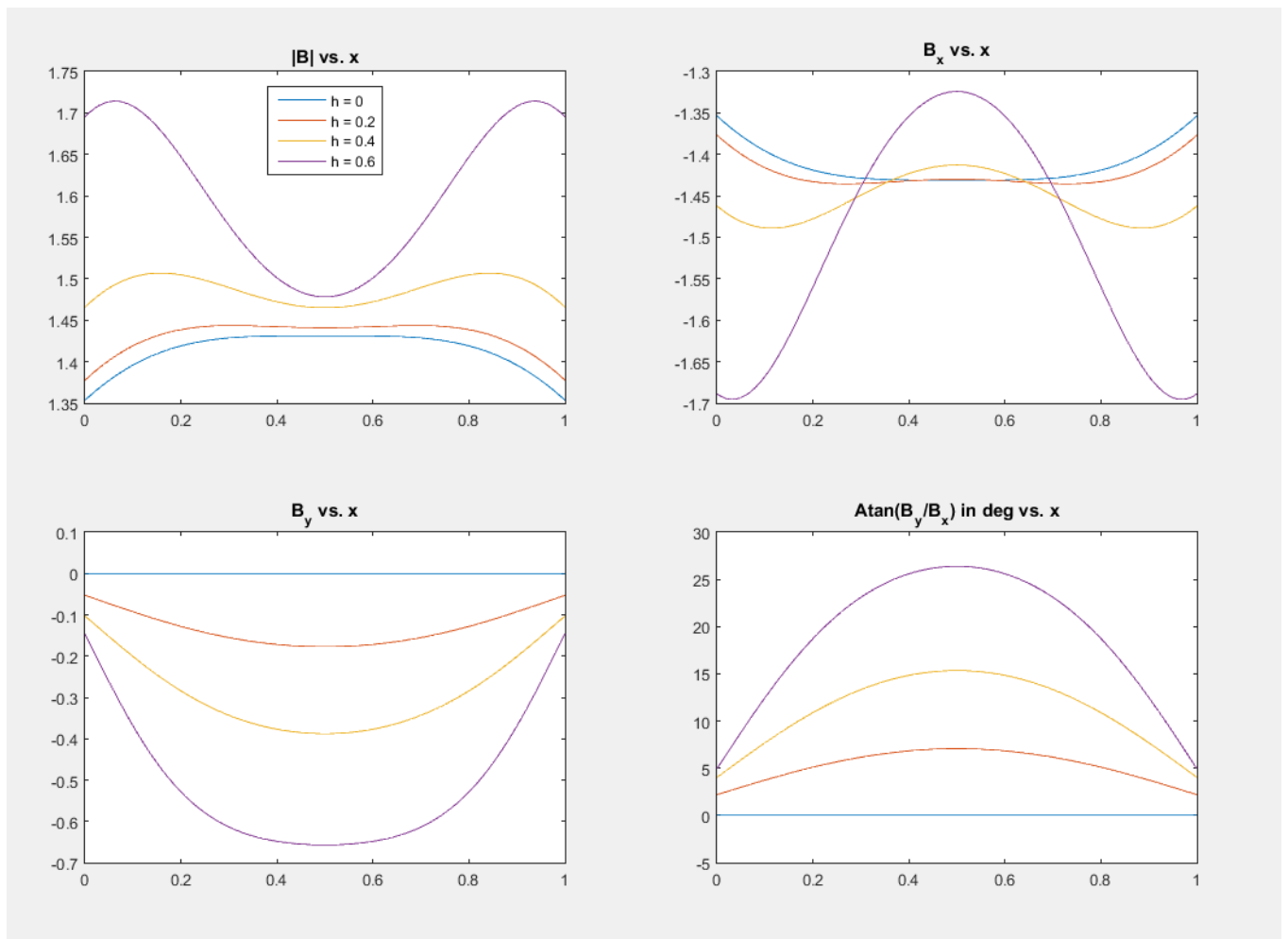


Figure 3