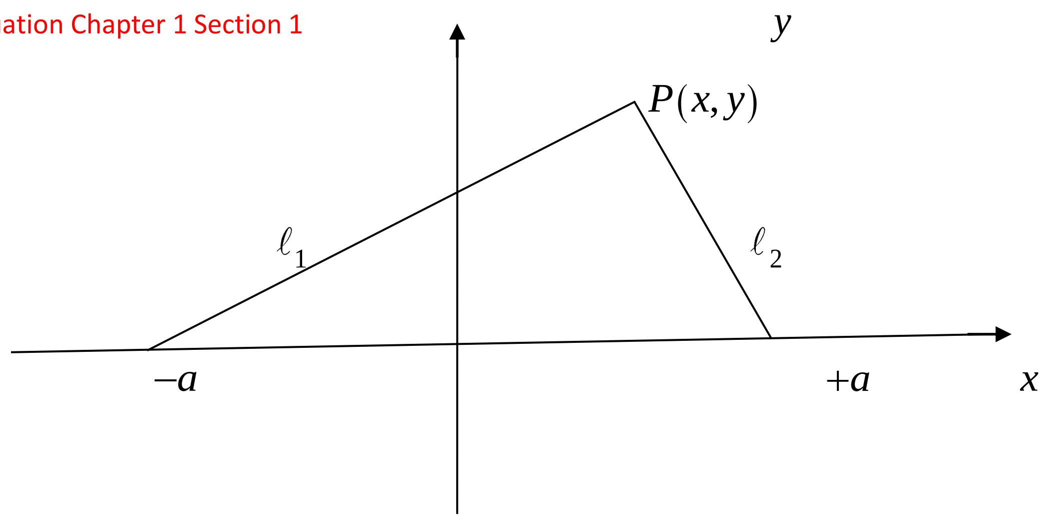
Equation Chapter 1 Section 1



To Prove: The point P traces out an ellipse if the lengths ℓ_1 and ℓ_2 from the points $x = -a$ and $x = +a$ add up to a constant.

Proof

Let

$$\ell_1 + \ell_2 = 2L > 2a \tag{1}$$

Pythagoras yields

$$\ell_1^2 = (a + x)^2 + y^2 = x^2 + 2ax + a^2 + y^2 \tag{2}$$

$$\ell_2^2 = (x - a)^2 + y^2 = x^2 - 2ax + a^2 + y^2 \tag{3}$$

Subtract eq. (3) from eq. (2):

$$\ell_1^2 - \ell_2^2 = (\ell_1 - \ell_2)(\ell_1 + \ell_2) = 4ax \tag{4}$$

Then

$$(\ell_1 - \ell_2) = 4ax / (\ell_1 + \ell_2) = 2ax / L \tag{5}$$

Solving for ℓ_1 and ℓ_2 from eqs (1) and (5) yields

$$\ell_1 = L + \frac{ax}{L} \tag{6}$$

$$\ell_2 = L - \frac{ax}{L} \qquad (7)$$

Equate the square of eq. (6) to eq. (2):

$$\left(L + \frac{ax}{L}\right)^2 = L^2 + 2ax + \frac{a^2x^2}{L^2} = x^2 + 2ax + a^2 + y^2 \qquad (8)$$

Collect terms to give

$$\left(L + \frac{ax}{L}\right)^2 = L^2 + 2ax + \frac{a^2x^2}{L^2}$$

$$\Rightarrow$$

$$x^2\left(1 - \frac{a^2}{L^2}\right) + y^2 = L^2 - a^2 = y_{\max}^2 \qquad (9)$$

Equation (9) is the equation of an ellipse with axes $2a$ and $2y_{\max} = 2\left(L^2 - a^2\right)^{1/2}$. If $a^2 > L^2/2$ then $2a$ is the major axis and similarly if $a^2 < L^2/2$ then $2a$ is the minor axis. If $a = 0$ then eq. (9) reduces to that of a circle with radius L.

QED